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Optimal Complexity Recovery of Band- and Energy-Limited Signals II

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This paper deals with the recovery of band- and energy-limited signals in $L_p(I)$-norm from Hermitian information gathered on a given finite interval I. Let $m_p(\varepsilon)$ be the minimal number of the information pieces required to find an ε-accurate approximation to any such signal. We shall prove that

$$\lim_{\varepsilon\to 0^+} \frac{m_p(\varepsilon)\log\log 1/\varepsilon}{\log 1/\varepsilon} = 1$$

for any p in $[1, \infty]$, and that for sufficiently small $\varepsilon > 0$, Hermitian interpolation using $m_p(\varepsilon)(1 + o(1))$ arbitrary nodes yields an ε-approximation in $L_p(I)$-norm with almost minimal cost. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider the class $J = J(\Omega_0)$ of signals $\check{X}$ of bandwidth $[-\Omega_0, \Omega_0]$,

$$\check{X}(t) = \int_{-\Omega_0}^{\Omega_0} X(\Omega)\exp(\dot{\ell}\,\Omega\, t)d\Omega \qquad (X \in L_2(-\Omega_0, \Omega_0),\ \dot{\ell} = \sqrt{-1}),$$

such that the energy of $\check{X}$ is bounded by 2π, i.e.,

$$\int_{-\infty}^{\infty} |\check{X}(t)|^2 dt \leq 2\pi.$$

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According to the Parseval theorem we have

$$J = \{\check{X}: X \in L_2(-\Omega_0, \Omega_0), \int_{-\Omega_0}^{\Omega_0} |X(\Omega)|^2 d\Omega \leq 1\}.$$

Our aim is to study the optimal complexity recovery of $\check{X} \in J$ by means of algorithms whose sole knowledge about $\check{X}$ is information $N(\check{X})$ consisting of n divided differences of $\check{X}$,

$$N(\check{X}) = [\check{X}[t_1], \check{X}[t_1, t_2], \ldots, \check{X}[t_1, t_2, \ldots, t_n]]^T, \tag{1}$$

where the points t_j belong to a given interval $I = [a - \tau, a + \tau]$ and satisfy the implication

$$t_j = t_{j+k} \Rightarrow \underset{l=1,2,\ldots,k}{\forall} t_j = t_{j+l} \qquad (j + k \leq n). \tag{2}$$

We deal with the worst-case setting using the $L_p(I)$-norm error criterion, $1 \leq p \leq \infty$. By an algorithm, we mean an arbitrary mapping ϕ: $N(J) \to L_p(I)$ and we measure the error $e_p(\phi)$ of ϕ by its worst performance in the class, J, with respect to $L_p(I)$-norm. Hence,

$$e_p(\phi) = \sup\left\{\left(\int_I |\check{X}(t) - \phi(N(\check{X}))(t)|^p dt\right)^{1/p} : \check{X} \in J\right\}.$$

Given a positive number ε, let $m_p(\varepsilon)$ denote the minimal number of points t_j required to find an algorithm ϕ^* acting on information vectors (1) and satisfying

$$e_p(\phi^*) \leq \varepsilon.$$

We call $\phi^*(N(\check{X}))(t)$ an (ε, p)-approximation to $\check{X}(t)$.

Throughout the paper we assume that the cost of arithmetic operations $(+, -, \times, /)$ and the cost of evaluating a signal at a point are taken as unity and **c**, respectively. Without loss of generality we also assume that the points t_j are nonadaptively (simultaneously) chosen (see Traub and Wozniakowski, 1980).

Let us define the functions v_k E $L_2(-\Omega_0, \Omega_0)$ by the $(k - 1)$st divided difference of the function $\exp(-\dot{\ell}\Omega\cdot)$, i.e.,

$$v_k(\Omega) = \exp(-\dot{\ell}\Omega\cdot)[t_1, t_2, \ldots, t_k], \qquad k = 1, 2, \ldots, n. \tag{3}$$

One can easily prove that for any $\check{X} \in J$ the divided differences of $\check{X}$ are values of some signals at zero. Namely, we have

$$\check{X}[t_1, t_2, \ldots, t_k] = \int_{-\Omega_0}^{\Omega_0} X(\Omega)\overline{v_k(\Omega)}d\Omega = \check{X}\bar{v}_k(0).$$

Thus, since the functions v_k can be precomputed, we see that $n\mathbf{c}$ can be taken as the cost of computing the information (1).

Let comp$_p(\varepsilon)$ denote the minimal computational cost of $\phi^*(N(\check{X}))(t)$.

The motivation of this paper goes back to a result of Kowalski (1986), which states that

$$\lim_{\varepsilon\to 0^+} \frac{m_\infty(\varepsilon) \log \log 1/\varepsilon}{\log 1/\varepsilon} = 1.$$

Since $L_p(I)$-norms become weaker when p decreases, it may be expected that the quantity

$$\lim_{\varepsilon\to 0^+} \frac{m_p(\varepsilon) \log \log 1/\varepsilon}{\log 1/\varepsilon}$$

decreases with p. In spite of this assertion we shall prove the following theorem:

THEOREM 1. *For any* $p \in [1, \infty]$ *we have*

(i) $\lim_{\varepsilon\to 0^+} (m_p(\varepsilon) \log \log(1/\varepsilon)/\log(1/\varepsilon) = 1,$

(ii) $\text{comp}_p(\varepsilon) = (\mathbf{c} + a)(\log(1/\varepsilon)/\log\log(1/\varepsilon))(1 + o(1))$ *as* $\varepsilon \to 0^+$, *where* $a \in [0, 3]$.

(iii) *For sufficiently small* $\varepsilon > 0$, *Hermitian interpolation using* log $(1/\varepsilon)(\log \log(1/\varepsilon)^{-1}(1 + o(1))$ *arbitrary nodes from* I *yields an* (ε, p)*-approximation with almost minimal cost.*

In section 4 we shall show that using results of Melkman (1977), a sharper theorem can be obtained in the case when $p = 2$.

2. AUXILIARY LEMMAS

Let $t_1, t_2, \ldots, t_n$ be arbitrary points on the interval $I = [a - \tau, a + \tau]$ and satisfying the condition (2). We denote by $G = G(v_1, v_2, \ldots, v_n)$ the Gram matrix $(\langle v_l, v_k\rangle)_{k,l=1}^n$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(-\Omega_0, \Omega_0)$ and v_l are the functions defined by (3).

The problem of optimal-error recovery of signals $\check{X} \in J(\Omega_0)$ from $N(\check{X}) = [\langle X, v_1\rangle, \langle X, v_2\rangle, \ldots, \langle X, v_n\rangle]^T$ can be stated as follows:

Find a mapping (algorithm) ϕ: $N(J) \to L_p(-\Omega_0, \Omega_0)$ that minimizes the worst-case error $e_p(\phi) = e_p(\phi: a, \tau, \Omega_0, t_1, \ldots, t_n)$, where

$$e_p(\phi) = \sup\left\{\left(\int_I |\check{X}(t) - \phi(N(\check{X}))(t)|^p dt\right)^{1/p} : X \in B(\Omega_0)\right\}$$

and where $B(\Omega_0)$ is the unit ball in $L_2(-\Omega_0, \Omega_0)$.

From this formulation it is easily seen that the problem is a particular case of that studied by Micchelli and Rivlin (1977).

Let us define:

$$r_p(t, \tau, \Omega_0; t_1, t_2, \ldots, t_n) = \sup\left\{\left(\int_I |\check{X}(t)|^p dt\right)^{1/p} : \check{X} \in J(\Omega_0\right.$$
$$\left.\check{X}[t_1] = \ldots = \check{X}[t_1, t_2, \ldots, t_n] = 0\right\}, \quad (4)$$

where we use the usual convention that

$$\left(\int_I |\check{X}(t)|^p dt\right)^{1/p} = \sup\{|\check{X}(t)|: t \in I\}$$

when $p = \infty$. Then clearly

$$r_p(a, \tau, \Omega_0; t_1, t_2, \ldots, t_n)$$
$$= \sup\left\{\left(\int_I |\check{X}(t)|^p dt\right)^{1/p} : X \in B(\Omega_0) \cap \operatorname{span}(v_1, v_2, \ldots, v_n)^\perp\right\}.$$

Then, by the result of Micchelli and Rivlin mentioned above, we have

$$r_p(a, \tau, \Omega_0; t_1, t_2, \ldots, t_n) = \inf_\phi e_p(\phi), \quad (5)$$

where the infimum is taken over all algorithms ϕ: $N(J) \to L_p(-\Omega_0, \Omega_0)$ and is attained by

$$\phi^*: \phi^*(N(\check{X})) = \sum_{k=1}^{n} \beta_k \check{v}_k, \quad (6)$$

where β_k is the kth component of the vector $G^{-1}N(\check{X})$.

To prove Theorem 1 we need a few auxiliary lemmas.

LEMMA 1. *For arbitrary numbers $p \in [1, \infty]$, $a \in \mathbf{R}$, τ, $\Omega_0 \in \mathbf{R}_+$ and arbitrary points $t_1, t_2, \ldots, t_n \in I = [a - \tau, a + \tau]$ satisfying* (2) *we have*

$$r_p(a, \tau, \Omega_0; t_1, t_2, \ldots, t_n) = r_p(0, 1, \Omega_0\tau, \nu_1, \nu_2, \ldots, \nu_n)\tau^{(1/p-1/2)},$$

where $\nu_k = (t_k - a)/\tau$ for $k = 1, 2, \ldots, n$.

Proof. Let X be an element of $B(\Omega_0) \cap \operatorname{span}(v_1, v_2, \ldots, v_n)^\perp$. Then, $\check{X} \in J(\Omega_0)$ and $\check{X}[t_1] = \check{X}[t_1, t_2] = \check{X}[t_1, t_2, \ldots, t_n] = 0$. Consequently, $\check{X}(t)$ coincide with the remainder term of Hermitian interpolation of $\check{X}$, i.e.,

$$\check{X}(t) = \int_{-\Omega_0}^{\Omega_0} X(\Omega)(\exp(\dot{\ell}\Omega\cdot)[t, t_1, \ldots, t_n]) \prod_{k=1}^{n} (t - t_k) d\Omega. \tag{7}$$

It can be easily verified by induction on n that

$$\begin{aligned}(\exp(\dot{\ell}\Omega\cdot)[t, t_1, t_2, \ldots, t_n]) \prod_{k=1}^{n} (t - t_k) \\ = \exp(\dot{\ell}\Omega a)(\exp(\dot{\ell}\Omega\tau\cdot)[\nu, \nu_1, \ldots, \nu_n]) \prod_{k=1}^{n} (\nu - \nu_k),\end{aligned}$$

where $\nu = (t - a)/\tau$. Thus, by substituting $\omega = \Omega\tau$ in (7) we get

$$\check{X}(\nu t + a) = \tau^{-1/2} \int_{-\Omega_0\tau}^{\Omega_0\tau} Y(\omega)(\exp(\dot{\ell}\omega\cdot)[\nu, \nu_1, \ldots, \nu_n]) \prod_{k=1}^{n} (\nu - \nu_k) d\omega,$$

where $Y(\omega) = \tau^{-1/2} X(\omega/\tau)\exp(\dot{\ell}\omega a/\tau)$. Therefore,

$$\begin{aligned}\left(\int_I |X(t)|^p dt\right)^{1/p} &= \tau^{1/p}\left(\int_{-1}^{1} |X(\nu\tau + a)|^p d\nu\right)^{1/p} \\ &= \tau^{(1/p - 1/2)}\left(\int_{-1}^{-1} \left| \int_{-\Omega_0\tau}^{\Omega_0\tau} Y(\omega)(\exp(\dot{\ell}\omega\cdot)[\nu, \nu_1, \ldots, \nu_n]) \right.\right. \\ &\quad \left.\left. \times \prod_{k=1}^{n} (\nu - \nu_k) d\omega \right|^p d\nu\right)^{1/p}.\end{aligned}$$

Since the operator $X \to Y$ maps the unit ball in $L_2(-\Omega_0, \Omega_0)$ onto the unit ball in $L_2(-\Omega_0\tau, \Omega_0\tau)$ and the function

$$\check{Y}(\nu) = \int_{-\Omega_0\tau}^{\Omega_0\tau} Y(\omega)\exp(\dot{\ell}\omega\nu) d\omega$$

has the property $\check{Y}[\nu_1] = \check{Y}[\nu_1, \nu_2] = \ldots = \check{Y}[\nu_1, \nu_2, \ldots, \nu_n] = 0$ we see that

$$\left(\int_I |\check{X}(t)|^p dt\right)^{1/p} = \tau^{(1/p - 1/2)}\left(\int_{-1}^{1} |\check{Y}(\nu)|^p d\nu\right)^{1/p}.$$

Hence, the statement of Lemma 1 follows. ■

Let us denote by $\|\cdot\|_p$ the norm in the space $L_p(-1, 1)$, $p \in [1, \infty]$.

LEMMA 2. *For arbitrary numbers $p \in [1, \infty]$, $\rho \in \mathbf{R}_+$ and arbitrary points $t_1, t_2, \ldots, t_n \in [-1, 1]$ satisfying* (2) *the following inequalities hold*:

$$2^{(1/p-1)}\, r_1(0, 1, \rho; t_1, t_2, \ldots, t_n) \leq r_p(0, 1, \rho; t_1, t_2, \ldots, t_n) \leq 2^{1/p}\, r_\infty(0, 1, \rho; t_1, \ldots, t_n).$$

Proof. The lemma is an immediate consequence of (4) and well-known inequalities

$$2^{(1/p-1)}\, \|\check{Y}\|_1 \leq \|\check{Y}\|_p \leq 2^{1/2}\|\check{Y}\|_\infty$$

that are valid for any Y in $L_2(-\rho, \rho)$. ∎

Given numbers $\rho \in \mathbf{R}_+$, $n \in \mathbf{N}$ and points $t_1, t_2, \ldots, t_n \in [-1, 1]$ let us define a number α and a complex-valued function f by the equations

$$\alpha = \rho/n \tag{8}$$

and

$$f(z) \equiv \left[\frac{\cos(\alpha z)}{z^2 - \pi^2/(4\alpha^2)}\right]^n \prod_{k=1}^{n} (z - t_k), \qquad z \in \mathbf{C}. \tag{9}$$

Let us also denote by $\|\cdot\|_{2,\infty}$ the norm in $L_2(-\infty, \infty)$, and by I_n the quantity

$$I_n = \int_{-\pi/2}^{\infty} \left(\frac{\sin x}{x}\right)^{2n} dx.$$

Using the formula $\ln(\sin x/x)^{2n} = 2n \sum_{k=1}^{\infty} \ln(1 - x^2/k^2\pi^2)$, $x \in (-\pi, \pi)$, one can prove that $I_n = \xi_n n^{n^{1/2}}/(n + \frac{1}{4})$, where $1.5 < \xi_n < 8.4$ and $\lim_{n\to+\infty} \xi_n = (3\pi)^{1/2}$.

We are now ready to present a lower bound on r_p.

LEMMA 3. *For any numbers $\rho \in \mathbf{R}_+$, $n \in \mathbf{N}$, and $p \in [1, \infty]$ and for arbitrary points $t_1, t_2, \ldots, t_n \in [-1, 1]$ satisfying* (2) *we have*

$$r_p(0, 1, \rho; t_1, t_2, \ldots, t_n) \geq (2\pi)^{1/2}\|f\|_p/\|f\|_{2,\infty}, \tag{10}$$

where f is the function defined by (8) *and* (9).

In particular, when $n \geq 2\rho/\pi$ the inequality (10) *yields*

$$r_1(0, 1, \rho; t_1, t_2, \ldots, t_n) \geq 2\left(\frac{\pi\rho}{nI_n}\right)^{1/2}\left(\frac{\rho}{2\pi n}\right)^n \tag{11}$$

and

$$r_2(0, 1, \rho; t_1, t_2, \ldots, t_n) \geq \pi \exp(-\tfrac{1}{24}) \left(\frac{\rho}{I_n(n + \frac{1}{2})}\right)^{1/2}\left(\frac{\rho}{2\pi n}\right)^n. \tag{12}$$

Proof. According to the Paley–Wiener theorem, $J(\rho)$ is precisely the class of entire functions g of exponential type ρ satisfying

$$\|g\|_{2,\infty} \leq (2\pi)^{1/2},$$

which are restricted to the real line (see Rudin, 1977).

Let us note now that the function f defined by (8) and (9) is an entire function of exponential type ρ which satisfies the equations

$$f[t_1] = f[t_1, t_2] = \ldots = f[t_1, t_2, \ldots, t_n] = 0.$$

Moreover f is square integrable over the real line and if $\alpha \leq \pi/2$, then

$$\begin{aligned}
\|f\|_{2,\infty}^2 &= \int_{-\infty}^{\infty} \left[\frac{\cos(\alpha x)}{x^2 - \pi^2/(4\alpha^2)}\right]^{2n} \prod_{k=1}^{n} (x - t_k)^2 dx \\
&= \alpha^{2n-1} \int_{-\infty}^{\infty} \left[\frac{\cos y}{(y - \pi/2)(y + \pi2)}\right]^{2n} \prod_{k=1}^{n} (y - \alpha t_k)^2 dy \\
&\leq \alpha^{2n-1} \int_{-\infty}^{\infty} \left[\frac{\cos y}{(y - \pi/2)(y + \pi/2)}\right]^{2n} (|y| + \alpha)^{2n} dy \\
&\leq 2\alpha^{2n-1} \int_0^{\infty} \left(\frac{\cos y}{y - \pi/2}\right)^{2n} dy = 2\alpha^{2n-1} \int_{-\pi/2}^{\infty} \left(\frac{\sin x}{x}\right)^{2n} dx \\
&= 2\alpha^{2n-1} I_n.
\end{aligned}$$

Consequently,

$$\|f\|_{2,\infty} \leq (2\alpha^{-1} I_n)^{1/2}\, \alpha^n. \tag{13}$$

when $n \geq 2\rho/\pi$.

Let us define the function g_0 to be the restriction to the real line of the complex mapping $z \to (2\pi)^{1/2} f(z)/\|f\|_{2,\infty}$. Then, of course,

$$g_0[t_1] = g_0[t_1, t_2] = \ldots = g_0[t_1, t_2, \ldots, t_n] = 0$$

and $g_0 \in J(\rho)$. Thus, by (4) we obtain the inequality

$$r_p(0, 1, \rho; t_1, t_2, \ldots, t_n) \geq \|g_0\|_p,$$

which coincides with (10). It is easily seen that

$$\inf\left\{\left|\frac{\cos y}{y^2 - \pi^2/4}\right| : |y| \leq \alpha\right\} \geq \inf\left\{\left|\frac{\cos y}{y^2 - \pi^2/4}\right| : |y| \leq \pi/2\right\} = \pi^{-1} \quad (14)$$

if $\alpha \leq \pi/2$. We recall that

$$\inf\left\{\left\| \prod_{k=1}^{n} (\cdot - \xi_k)\right\|_p : \xi_k \in \mathbf{C}\right\} = \begin{cases} 2^{-(n-1)} & \text{when } p = 1, \\ \dfrac{2^n n! n!}{(n + \frac{1}{2})^{1/2}(2n)!} & \text{when } p = 2 \end{cases} \quad (15)$$

(see Timan, 1963). Using (14), (15), and the inequality $n \geq 2\rho/\pi$ we get

$$\begin{aligned} \|f\|_p &= \alpha^{2n}\left(\int_{-1}^{1} \left|\frac{\cos(\alpha x)}{(\alpha x)^2 - \pi^2/4}\right|^{np} \prod_{k=1}^{n} |x - t_k|^p dx\right)^{1/p} \\ &\geq \alpha^{2n} \inf\left\{\left| \frac{\cos y}{y^2 - \pi^2/4}\right| : |y| \leq \alpha\right\}^n \left\| \prod_{k=1}^{n} (\cdot - t_k)\right\|_p \qquad (16) \\ &\geq \alpha^{2n}\pi^{-n} \left\| \prod_{k=1}^{n} (\cdot\ t_k)\right\|_p \geq \begin{cases} 2\alpha^n(\alpha/2\pi)^n & \text{when } p = 1, \\ \dfrac{\alpha^n(2\alpha/\pi)^n n! n!}{(n + \frac{1}{2})^{1/2}(2n)!} & \text{when } p = 2. \end{cases} \end{aligned}$$

By the Stirling formula, $k! = (2\pi k)^{1/2}(k/e)^n \exp(\delta/12k)$, $0 < \delta < 1$, we have

$$\frac{n! n!}{(2n)!} \geq \exp(-\tfrac{1}{24})(\pi n)^{1/2} 2^{-2n}.$$

This inequality, taken together with (10), (13), and (16) finally yields

$$r_p(0, 1, \rho; t_1, t_2, \ldots, t_n) \geq \begin{cases} 2(\pi\alpha/I_n)^{1/2}(\alpha 2\pi)^n & \text{if } p = 1, \\ \pi \exp(-\frac{1}{24}) \left(\dfrac{\alpha n}{I_n(n + \frac{1}{2})}\right)^{1/2} (\alpha/2\pi)^n & \text{if } p = 2 \end{cases}$$

which gives (11) and (12) and which completes the proof. ∎

Let us denote by $H_n = H_n(N(\check{X}))$ the Hermite interpolatory algorithm that uses information $N(\check{X})$ defnied by (1) and (2), i.e.,

$$H_n(N(\check{X})(t) = \sum_{k=1}^{n} \check{X}[t_1, t_2, \ldots, t_k] \prod_{j=1}^{k-1} (t - t_j). \tag{17}$$

We are now in a position to find an upper bound on $r_\infty(0, 1, \rho; t_1, t_2, \ldots, t_n)$.

LEMMA 4. *For any numbers* $\rho \in \mathbf{R}_+$, $n \in \mathbf{N}$, *and arbitrary nodes* $t_1, t_2, \ldots, t_n \in [-1, 1]$ *satisfying* (2) *we have*

$$r_\infty(0, 1, \rho; t_1, t_2, \ldots, t_n) \leq e_\infty(H_n) < \frac{\rho^{1/2}}{(2\pi)^{1/2} n} \left(\frac{2e\rho}{n}\right)^n,$$

where $e_\infty(H_n) = e_\infty(H_n; 0, 1, \rho, t_1, t_2, \ldots, t_n)$.

Proof. The first inequality is an immediate consequence of (5). To prove the second inequality we proceed as in the proof of Lemma 5 of Kowalski (1986).

For any $X \in B(\rho)$ we have

$$\check{X}(t) - H_n(N(\check{X}))(t) = \check{X}[t, t_1, \ldots, t_n] \prod_{j=1}^{n} (t - t_j)$$

and

$$\check{X}[t, t_1, \ldots, t_n] = \int_{-\rho}^{\rho} X(\Omega)(\exp \dot{\ell}\, \Omega \cdot)[t, t_1, \ldots, t_n] d\Omega.$$

Therefore,

$$\begin{aligned} e_\infty(H_n) &= \sup\{|\check{X}(t) - H_n(N(\check{X}))(t)|: X \in B(\rho), t \in [-1, 1]\} \\ &= \sup\Big\{\prod_{j=1}^{n} |t - t_j| \sup_{X \in B(\rho)} \Big| \int_{-\rho}^{\rho} X(\Omega)(\exp(\dot{\ell}\, \Omega \cdot)[t, t_1, \ldots, t_n]) \\ &\qquad d\Omega\Big|: t \in [-1, 1]\Big\} \\ &= \sup\Big\{\prod_{j=1}^{n} |t - t_j| \left(\int_{-\rho}^{\rho} |\exp(\dot{\ell}\, \Omega \cdot)[t, t_1, \ldots, t_n]|^2 d\Omega\right)^{1/2}: \\ &\qquad t \in [-1, 1]\Big\}. \end{aligned}$$

Since

$$|\exp(\acute{\ell}\,\Omega\cdot)[t, t_1, \ldots, t_n]| \leq |\Omega|^n/n!$$

and

$$\sup\left\{\prod_{j=1}^{n} |t - t_j|;\ t \in [-1, +1]\right\} \leq 2^n$$

we get

$$e_\infty(H_n) \leq \frac{2^n}{n!}\left(\int_{-\rho}^{\rho} \Omega^{2n} d\Omega\right)^{1/2} = \frac{2^n}{n!}\frac{\rho^{n+1/2}}{(n+\frac{1}{2})^{1/2}}.$$

Applying the Stirling formula we finally obtain

$$e_\infty(H_n) < \frac{\rho^{1/2}}{(2\pi)^{1/2}n}\left(\frac{2e\rho}{n}\right)^n,$$

which completes the proof. ■

Let us now mention that the Remarks (1)–(3) on the properties of Lagrangian interpolatory algorithms presented by Kowalski (1986) also apply to the Hermitian case. Namely, they can be restated as follows.

Remarks. (1) For $n > 2e\Omega_0\tau$, $H_n(N(\check{X}))(t)$ provides an exponentially good fit to $\check{X}(t)$ for any $\check{X} \in J(\Omega_0)$ and any $t \in I$, regardless of the nodes $t_1, t_2, \ldots, t_n \in I$ chosen; as is obvious from the inequality

$$e_\infty(H_n; a, \tau, \Omega_0, t_1, t_2, \ldots, t_n) < \frac{\Omega_0^{1/2}}{(2\pi)^{1/2}n}\left(\frac{2e\Omega_0\tau}{n}\right)^n.$$

In practice, when nonexact information is available, good approximation of the signal is possible for at most a bounded distance d beyond the interval of observation $I_1 = \operatorname{conv}(t_1, t_2, \ldots, t_n)$, where d is independent of n (see Landau, 1985).

(2) Much better estimates on $e_\infty(H_n; a, \tau, \Omega_0, t_1, t_2, \ldots, t_n)$ are possible for special choices of nodes $t_1, t_2, \ldots, t_n$. For example, if t_k are the Chebyshev points in I, i.e., if

$$t_k = t_k^* = a + \tau \cos\frac{(k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n,$$

then

$$e_\infty(H_n;\, a,\, \tau,\, \Omega_0,\, t_1,\, t_2,\, \ldots,\, t_n) < \frac{2\Omega_0^{1/2}}{(2\pi)^{1/2}n}\left(\frac{e\Omega_0\tau}{2n}\right)^n.$$

(3) For any points $t_1, t_2, \ldots, t_n \in I$ satisfying (2), we have

$$e_\infty(H_n;\, a,\, \tau,\, \Omega_0,\, t_1,\, \ldots,\, t_n) \geq (2\Omega_0)^{1/2}\mathbf{e}_{n-1},$$

where $\mathbf{e}_{n-1}$ is the error of the uniform approximation of the function $\sin(\Omega_0\tau\cdot)/\Omega_0\tau\cdot$ on $[-1, 1]$ by its Hermite polynomial associated with the nodes $\nu_1, \nu_2, \ldots, \nu_n$, where $\nu_k = (t_k - a)/\tau$.

The right-hand side of the last inequality is at least of order 0.1 $((\Omega_0\tau/2)^{-1/2}$ when $n < n_0 \approx (\frac{10}{11})\Omega_0\tau$.

3. Proof of Theorem 1

Let us select arbitrary numbers $p \in [1, \infty]$, $a \in \mathbf{R}$, τ, $\Omega_0 \in \mathbf{R}_+$ and arbitrary points $t_1, t_2, \ldots, t_n \in I = [a - \tau, a + \tau]$. Then, for $n \geq 2\Omega_0\tau/\pi$, Lemmas 1, 2, 3, 4 yield

$$\begin{aligned} k_1 n^{-1/4}(k_2 n)^{-n} &< \tau^{(1/p-1/2)}\, 2^{(1/p-1)}\, r_1(0,\, 1,\, \Omega_0,\, \tau;\, \nu_1,\, \ldots,\, \nu_n) \\ &\leq r_p(a,\, \tau,\, \Omega_0;\, t_1,\, \ldots,\, t_n) \\ &\leq \tau^{(1/p-1/2)}\, 2^{1/p} r_\infty(0,\, 1,\, \Omega_0\tau;\, \nu_1,\, \ldots,\, \nu_n) < k_3 n^{-1}\, (k_4 n)^{-n}, \end{aligned}$$

where $\nu_j = (t_j - a)/\tau$ $(j = 1, 2, \ldots, n)$ and where

$$\begin{aligned} k_1 &= \tau^{(1/p-1/2)}\, 2^{1/p}(\pi\Omega_0\tau/8.4)^{1/2}, \\ k_2 &= 2\pi/(\Omega_0\tau), \\ k_3 &= (2\tau)^{1/p}(2\pi)^{-1/2}\Omega_0^{1/2}, \\ k_4 &= 1/(2e\Omega_0\tau). \end{aligned}$$

Thus, for sufficiently small positive ε, the number $m_p(\varepsilon)$ satisfies

$$k_1 m_p(\varepsilon)^{-1/4}(k_2 m_p(\varepsilon))^{-m_p(\varepsilon)} < \varepsilon < k_3 m_p(\varepsilon)^{-1}\, (k_4 m_p(\varepsilon))^{-m_p(\varepsilon)}. \tag{18}$$

From this one easily concludes that

$$m_p(\varepsilon) = \frac{\log 1/\varepsilon}{\log \log 1/\varepsilon}(1 + o(1)) \qquad \text{as } \varepsilon \to o$$

which proves (i).

Now let $m_p'(\varepsilon)$ denote the minimal number of nodes $t_1, t_2, \ldots, t_n \in I$ satisfying (2) and such that the corresponding Hermite algorithm H_n gives an (ε, p)-approximation, independent of their location in I. That is,

$$m_p'(\varepsilon) = \min\{m: e_p(H_n; a, \tau, \Omega_0, t_1, \ldots, t_m) \\ \leq \varepsilon, \forall t_1, t_2, \ldots, t_m \in I \rightleftarrows \text{sat.}(2)\}.$$

From Remark 1 we conclude that $e_p(H_n; a, \tau, \Omega_0, t_1, \ldots, t_n) = \Theta(1/n) (k_4 n)^n)$ which proves that

$$m_p'(\varepsilon) = m_p(\varepsilon)(1 + o(1)) = \frac{\log 1/\varepsilon}{\log \log 1/\varepsilon}(1 + o(1)) \qquad \text{as } \varepsilon \to 0^+.$$

Let $\text{comp}_p(H_{m'(\varepsilon)})$ denote the cost of producing an (ε, p)-approximation to $\check{X}(t)(\check{X} \in J(\Omega_0), t \in I)$ by the Hermite algorithm (17) that uses $m_p'(\varepsilon)$ nodes. By applying the Horner scheme to the right-hand side of (17) with $n = m_p'(\varepsilon)$ we see that $\text{comp}_p(H_n)$ consists of the cost of $m_p'(\varepsilon)$ evaluations of the divided differences and the cost of $3(m_p'(\varepsilon) - 1)$ arithmetic operations. Thus,

$$\text{comp}_p(H_{m'(\varepsilon)}) \leq (\mathbf{c} + 3)m_p'(\varepsilon) - 3. \tag{20}$$

On the other hand we have

$$\mathbf{c}m_p(\varepsilon) \leq \text{comp}_p(\varepsilon) \leq \text{comp}_p(H_{m'(\varepsilon)}). \tag{21}$$

Let us finally note that the inequalities (20) and (21) taken together with (19) imply that the quantities $\text{comp}_p(\varepsilon)$ and $\text{comp}_p(H_{m'(\varepsilon)})$ both behave as

$$(\mathbf{c} + a)\frac{\log 1/\varepsilon}{\log \log 1/\varepsilon}(1 + o(1)) \qquad \text{when } \varepsilon \to 0^+.$$

This gives (ii) and (iii) and completes the proof.

4. The Case $p = 2$

Let us note that the main result of this paper can be sharpened in the case when $p = 2$. To this end, we recall a definition and basic properties of the prolate spheroidal wave functions ψ_k.

It is well known that the integral operator

$$L_2(-\tau,\tau) \ni f \to \mathcal{K}f, \qquad \mathcal{K}f(t) = \int_{-\tau}^{\tau} \exp\left(\frac{i\,\Omega_0 tx}{\tau}\right) f(x)dx$$

has the eigenvalues α_k ($k = 0, 1, 2, \ldots$) satisfying

$$|\alpha_0| > |\alpha_1| > |\alpha_2| > \ldots \lim_{k\to+\infty}|\alpha_k| = 0$$

and the eigenfunctions (prolate spheroidal wave functions) ψ_k,

$$\mathcal{K}\psi_k = \alpha_k\psi_k, \tag{22}$$

that are *real band-limited to the level* Ω_0 *and orthonormal in* $L_2(-\infty, \infty)$. Moreover, we have

(i) $$\int_{-\tau}^{\tau} (\sin(\Omega_0(t-s))/\pi(t-s)\ \psi_k(s)ds = \lambda_k\psi_k(t),$$

where λ_k depends on the product $\Omega_0\tau$ only and satisfies

$$\lambda_k = |\alpha_k|^2\Omega_0/(2\tau\pi), \qquad k = 0, 1, \ldots.$$

(ii) $$\int_{-\tau}^{\tau} \psi_k(t)\psi_l(t)dt = \delta_{kl}\lambda_k \qquad (k, l = 0, 1, \ldots).$$

(iii) The set $\{\psi_k\}_{k=0}^{\infty}$ is complete in $L_2(-\tau, \tau)$ and in $J(\Omega_0)$.

(iv) For each k, the function ψ_k has exactly k simple zeros $\xi_{k,1}$, $\xi_{k,2}, \ldots, \xi_{k,k}$ in the interval $(-\tau, \tau)$.

For the properties of the functions ψ_k mentioned above see Slepian and Pollak (1961) and Slepian (1965). More information about ψ_k and λ_k can be found in the papers by Landau and Pollak (1961, 1962).

Let us now consider the problem of optimal recovery of $\check{X} \in J(\Omega_0)$ and $L_2(I)$-norm by algorithms that use information of the form

$$\mathbf{N}_n(\check{X}) = [L_1(\check{X}), L_2(\check{X}), \ldots, L_n(\check{X})]^T,$$

where L_k are linear functionals on $L_2(-\Omega_0, \Omega_0)$. Melkman (1977) solved this problem by showing that

$$\inf_{\mathbf{N}_n}\ \inf_{y:\mathbf{N}_n(J)\to L_2(I)}\ \sup_{X\in J} \left(\int_I |\check{X}(t) - y(\mathbf{N}_n(\check{X}))(t)|^2dt\right)^{1/2} = (2\pi\lambda_n)^{1/2}. \tag{23}$$

He also showed that the infima in (23) are attained by the information operator

$$N_n(\check{X}) + [\check{X}(s_1), \check{X}(s_2), \ldots, \check{X}(s_n)]^T$$

and the algorithm

$$\Phi(N_n(\check{X})) = \sum_{k=1}^{n} \gamma_k \check{u}_k,$$

where γ_k is the kth component of the vector $G(u_1, u_2, \ldots, u_n)^{-1} N_n(\check{X})$ and where $u_k(t) \equiv \exp(-\dot{\ell}(a + \xi_{n,k})t)$ for $k = 1, 2, \ldots, n$. Since $N_n(\check{X})$ is equivalent to information (1) with $t_k = a + \xi_{n,k}$ $(k = 1, 2, \ldots, n)$, $m_2(\varepsilon)$ in *Theorem* 1 *for* $p = 2$ *can be redefined to mean the minimal number of linear functional evaluations required to find an* $(\varepsilon, 2)$*-approximation to any* $\check{X} \in J(\Omega_0)$.

Let $(\cdot, \cdot)$ denote the inner product in $L_2(I)$ and let

$$f_k(s) \equiv \lambda_k^{-1/2}\psi_k(s - a) \qquad \text{for } k = 1, 2, \ldots, n.$$

Melkman (1977) showed that another choice of $\mathbf{N}_n$ and ϕ attaining the infima in (23) is $\mathbf{N}_n = \mathcal{N}_n$ and $\phi = \Phi^*$, where

$$\mathcal{N}_n(\check{X}) = [(\check{X}, f_1), (\check{X}, f_2), \ldots, (\check{X}, f_n)]^T$$

and where

$$\Phi^*(\mathcal{N}_n(\check{X})) = \sum_{k=0}^{n-1} (\check{X}, f_k) f_k. \tag{24}$$

For any $X \in L_2(-\Omega_0, \Omega_0)$ and any $s \in I$ we have

$$(\check{X}, f_k) = \left(\int_{-\Omega_0}^{\Omega_0} X(\Omega)\exp(\dot{\ell}\,\Omega\cdot)d\Omega,\ f_k\right) = \int_{-\Omega_0}^{\Omega_0} X(\Omega) y_k(\Omega) d\Omega = \check{X} y_k(0),$$

where $y_k(\Omega) = \int_I \exp(\dot{\ell}\,\Omega t) f_k(t) dt$. Moreover, (22) implies that

$$\begin{aligned} f_k(s) &= \lambda_k^{-1/2}\alpha_k^{-1}(\mathcal{H}\psi_k)(s - a) \\ &= \int_{-\Omega_0}^{\Omega_0} \exp(\dot{\ell}\,\Omega s)\delta_k(\Omega)d\Omega = \check{\delta}_K(s), \end{aligned}$$

where $\delta_k(\Omega) = \tau\lambda_k^{-1/2}(\alpha_k\Omega_0)^{-1}\psi_k(\Omega\tau/\Omega_0)$. Consequently, (24) can be rewritten in the form

$$\Phi^*(\mathcal{N}_n(X)) = \sum_{k=0}^{n-1} \check{X}y_k(0)\check{\delta}_k.$$

Since the functions y_k can be precomputed, we need $2n$ measurements of the signals' values $\check{X}y_k(0)$, $\check{\delta}_k(s)$ and $2n - 1$ arithmetic operations in order to compute $\Phi^*(\mathcal{N}_n(\check{X}))(s)$. When $n = m_2(\varepsilon)$ we see that the optimal error algorithm (24) yields an $(\varepsilon, 2)$-approximation with *almost minimal cost* $2(\mathbf{c} + 1)m_2(\varepsilon) - 1$.

The Melkman results, mentioned above, taken together with Lemmas 1–3 and Remark 2, imply that for sufficiently large n we have

$$\exp(-\tfrac{1}{12}) \frac{\pi\Omega_0\tau}{2I_n(n + \frac{1}{2})} \left(\frac{\Omega_0\tau}{2\pi n}\right)^{2n} < \lambda_n < \frac{2\Omega_0\tau}{(\pi n)^2} \left(\frac{e\Omega_0\tau}{2n}\right)^{2n},$$

which improves the Slepian's result: $\lambda_n = 0((e\Omega_0\tau/\pi n)^{2n})$ as $n \to +\infty$ (see Landau, 1965).

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